

Quantum propagators in a random metric

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Abstract

We consider second order differential operators with coefficients which are Gaussian random fields. When the covariance becomes singular at short distances then the propagators of the Schrödinger equation as well as of the wave equation behave in an anomalous way. In particular, the Feynman propagator for the wave equation is less singular than the one with deterministic coefficients. We suggest some applications to quantum gravity.

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Diffusions generated by second order differential operators with random coefficients have been studied for some time. Randomness changes the long time behaviour of fluctuations. However, the short time behaviour is extremely stable with respect to perturbations. As an example, in a regular random field the mean square displacement for a small time is always linear in time. As a consequence, the mean value of the Green's function (the resolvent) in d -dimensional space behaves as $|\mathbf{x} - \mathbf{y}|^{2-d}$ when the distance $|\mathbf{x} - \mathbf{y}|$ tends to zero. Solutions of the Schrödinger equation can usually be obtained from the solutions of the diffusion equation by an analytic continuation in time. Hence, an analogous behaviour is expected.

A long time ago it has been suggested by Pauli (see [1]-[2]) that quantization of the gravitational field can remove divergencies of the conventional quantum field theory. The divergencies arise as a consequence of the singular behaviour of the Green's functions. Hence, if the Green's functions averaged over the metric fluctuations were regular then the divergencies of the quantum field theory would not appear at all. Some perturbative calculations have been performed recently, see [4] -[5], showing that there are no singularities on the light cone.

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In this letter we study a simple model with a random Gaussian metric. We discuss first the Schrödinger equation with a random Hamiltonian. We consider a singular metric. After a renormalization the mean value of the Feynman propagator of the Schrödinger equation shows an anomalous behaviour for a small time. Then, by means of the proper time method [3] we can obtain the Green's function for the wave operator (the Feynman's causal propagator). We show that (if $d \leq 4$) the expectation values of the Green's functions are more regular at short distances than the ones with a deterministic metric (however our exact results do not confirm heuristic arguments in refs.[4]-[5] which are based on approximate calculations).

We consider the Schrödinger equation in R^{2n}

$$i\partial_\tau\psi = H\psi = -\frac{1}{2}\sum_{k=1}^{2n} : \alpha_k^2 : \partial_k^2\psi \quad (1)$$

where α_k are independent homogeneous Gaussian random fields (depending only on one coordinate of the vector $\mathbf{x} \in R^{2n}$) with the same covariance

$$\langle \alpha_k(x)\alpha_k(y) \rangle = G(x-y) \quad (2)$$

The Hamiltonian H in (1) is symmetric in $L^2(d\mathbf{x})$ because we choose α_k independent of x_k . We begin with a regularized field α with a regular covariance G . Then both terms in $:\alpha^2:(x) = \alpha^2(x) - \langle \alpha^2(x) \rangle = \alpha^2(x) - G(0)$ are well-defined. Subsequently, the Wick square of a generalized random field α is defined as a limit when the regularization is removed.

We solve the Schrödinger equation (1) with the initial condition ψ by means of a probabilistic version of the Feynman integral [7]-[8]

$$\psi_\tau(\mathbf{x}) = E[\psi(\mathbf{q}_\tau(\mathbf{x}))] \quad (3)$$

where $E[.]$ is an expectation value with respect to the Brownian motion defined as the Gaussian process on R^{2n} with the covariance

$$E[b_j(t)b_k(s)] = \delta_{jk} \min(s, t)$$

In order to write the solution of the Schrödinger equation (1) in the form (3) we assume that ψ is an analytic function. Then, we begin with regularized random fields which are analytic functions as well. We remove the regularization only after a calculation of the expectation values over α . The process $\mathbf{q}_\tau(\mathbf{x})$ starts from \mathbf{x} at $\tau = 0$. It is a solution of a set of stochastic equations [6]. In order to write down the solution explicitly we set $\alpha_{2k} = 1$ and $\alpha_{2k-1}(\mathbf{x}) = \alpha_{2k-1}(x_{2k})$ for $k = 1, \dots, n$. Then,

$$q_{2k}(\tau, \mathbf{x}) = x_{2k} + \lambda b_{2k}(\tau) \quad (4)$$

and the odd components of the process can be expressed by the Ito integral

$$q_{2k-1}(\tau, \mathbf{x}) = x_{2k-1} + \lambda \int_0^\tau \alpha_{2k-1}(q_{2k}(s, \mathbf{x})) db_{2k-1}(s) \quad (5)$$

where $\tau \geq 0$ and

$$\lambda = \sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$$

It can be checked by direct differentiation using the Ito calculus [6] that ψ_τ solves the Schrödinger equation (1) for each (regularized) α . If we represent ψ in terms of its Fourier transform then we can calculate explicitly the average over α of the products

$$\langle \psi_{\tau_1}(\mathbf{x}_1) \dots \overline{\psi}_{s_1}(\mathbf{y}_1) \dots \rangle$$

We restrict ourselves to the expectation value of the Feynman propagator $K_\tau(\mathbf{x}, \mathbf{y})$. The time evolution is determined by the propagator

$$\psi_\tau(\mathbf{x}) = \int K_\tau(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) d\mathbf{y}$$

We further specify the random fields α_k in eq.(1) by assuming that their covariance is of the form

$$G(x) = \int_{-\infty}^{\infty} d\nu \rho(\nu) \cos(\nu x^4) \quad (6)$$

Then, $G(\sqrt{i}x) = G(x)$. For the computation of $\langle K_\tau \rangle$ it is sufficient to insert into eq.(3) $\exp(-i\mathbf{p}\mathbf{y})$ as the Fourier transform of $\psi(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})$ and to note that the Fourier transform of the r.h.s. of eq.(3) is $\exp(-i\mathbf{p}\mathbf{y} - i\mathbf{p}\mathbf{x} + i\mathbf{p}\mathbf{q}_\tau(\mathbf{x}))$. The computation of the Gaussian integral over α and the momentum integrals p_{2k-1} is elementary. As a result we obtain the following formula

$$\begin{aligned} \langle K_\tau(\mathbf{x}, \mathbf{y}) \rangle &= (2\pi)^{-n} E[\int \prod_{k=1}^n dp_{2k} \exp(ip_{2k}(x_{2k} - y_{2k}) + i\lambda p_{2k} b_{2k}(\tau)) \\ &\quad (2\pi\Gamma_k(\tau))^{-\frac{1}{2}} \exp\left(\frac{i}{2}(x_{2k-1} - y_{2k-1})^2 / \Gamma_k(\tau)\right)] \end{aligned} \quad (7)$$

where

$$\Gamma_k(\tau) = 2 \int_0^\tau db_{2k-1}(t) \int_0^t db_{2k-1}(s) G(b_{2k}(t) - b_{2k}(s)) \quad (8)$$

In eq.(7) we have considered first a regularized G . The Gaussian integral over α gives the double integral $\int dbdbG$. Then, we obtain $\Gamma = \int dbdbG - G(0)\tau$ in eq.(7) ($G(0)$ comes from the Wick square in eq.(1)). The replacement of the double (unordered) integral by a time-ordered one leads to the formula (8).

We can calculate explicitly some "moments" of K , e.g.,

$$\langle x_{2k-1}^{2r} \rangle = \int d\mathbf{y} y_{2k-1}^{2r} \langle K_\tau(0, \mathbf{y}) \rangle = E[\Gamma_k^{2r}] \quad (9)$$

Under quite general assumptions the kernel K_τ of a regular second order differential operator behaves for a small time as $\tau^{-n} \exp(\frac{i}{\tau} A(\mathbf{x})|\mathbf{x} - \mathbf{y}|^2)$. Hence, for a small time $\langle x^{2r} \rangle \simeq \tau^r$. We show that such a behaviour fails in a singular random field.

In order to simplify the subsequent discussion we choose $\rho(\nu)$ in G (eq.(6)) in a scale invariant form

$$\rho(\nu) = \text{const} |\nu|^{\frac{\gamma}{2}-1}$$

where $\gamma < \frac{1}{2}$ if the stochastic integrals in eq.(8) are to make sense. Then

$$G(x) = |x|^{-2\gamma} \quad (10)$$

The integral (9) could be calculated explicitly but this is not necessary. We can use the scale invariance of the Brownian motion (saying that $\sqrt{cb}(t/c)$ has the same probability distribution as $b(t)$) in order to conclude that for an integer r

$$\langle x_{2k-1}^{2r} \rangle = A_r \tau^{r(1-\gamma)} \quad (11)$$

with certain constants A_r (here $A_1 = 0$). The even coordinates are described by a simple formula for the "moments" $\langle x_{2k}^{2r} \rangle = E[(b_{2k}(\tau))^{2r}] = C_r \tau^r$.

We did not use an assumption that α_k^2 in eq.(1) have the same sign, i.e., we could take some of $\alpha_k = i\tilde{\alpha}_k$ where $\tilde{\alpha}_k$ has the covariance (2). Let us assume that α_1 is purely imaginary, then $\Gamma_1 \rightarrow -\Gamma_1$ in eq.(7). We obtain on the r.h.s. of eq.(1) the operator (which is symmetric in $L^2(dx)$)

$$\mathcal{A} = \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \quad (12)$$

with the metric g which has the Minkowski signature $(\eta_{\mu\nu}) = (1, -1, -1, \dots, -1)$. This is the ghost operator in quantum gravity (in a particular gauge). We can obtain the causal Feynman propagator of \mathcal{A} (the Green's function) by means of the proper time method [3]

$$\mathcal{A}^{-1}(x, y) = i \int_0^\infty d\tau (\exp(-i\tau \mathcal{A}))(x, y) \quad (13)$$

The Green's function depends only on $x - y$. It cannot be calculated explicitly. However, we obtain a simple formula if we integrate over the even coordinates

$$\int \prod_k dx_{2k} \langle \mathcal{A}^{-1}(x, y) \rangle = i \int_0^\infty d\tau E[\prod_k (2\pi \Gamma_k(\tau))^{-\frac{1}{2}} \exp\left((-1)^{e_k} \frac{i}{2} (x_{2k-1} - y_{2k-1})^2 / \Gamma_k(\tau)\right)] \quad (14)$$

where $(-1)^{e_k} = \pm 1$ is the signature of the metric in eq.(12). Let us denote

$$(x - y)^2 = \sum_{k=1}^n \left((-1)^{e_k} (x_{2k-1} - y_{2k-1})^2 + (x_{2k} - y_{2k})^2 \right) \equiv (x - y)_o^2 + (x - y)_e^2$$

Then, changing the integration variable $\tau \rightarrow ((x - y)_o^2)^{\frac{1}{1-\gamma}} \tau$ we can conclude that at short distances

$$\int \prod_k dx_{2k} \langle \mathcal{A}^{-1}(x, y) \rangle \simeq ((x - y)_o^2)^{-\frac{n}{2} + \frac{1}{1-\gamma}} \quad (15)$$

The integral (15) is less singular than the one in the deterministic case which corresponds to $\gamma = 0$.

An estimate of the Green's function $\langle \mathcal{A}^{-1}(x, y) \rangle$ when $x \rightarrow y$ is simple if $x \rightarrow y$ either on the hyperplane $(x - y)_e^2 = 0$ or on the one with $(x - y)_o^2 = 0$. In eq.(7) we can write $b_{2k}(\tau) = \sqrt{\tau} b_{2k}(1)$ and change the integration variable $\sqrt{\tau} p_{2k} = p'_{2k}$. Then, in the first case by rescaling $\tau \rightarrow \left((x - y)_o^2\right)^{\frac{1}{1-\gamma}} \tau$ we obtain for short distances

$$\langle \mathcal{A}^{-1}(x, y) \rangle \simeq \left((x - y)_o^2\right)^{-\frac{n}{2} + (1 - \frac{n}{2})/(1-\gamma)} \quad (16)$$

In the case $(x - y)_o^2 = 0$ the rescaling $\tau \rightarrow (x - y)_e^2 \tau$ gives the result

$$\langle \mathcal{A}^{-1}(x, y) \rangle \simeq \left((x - y)_e^2\right)^{-\frac{n}{2} + 1 - \frac{n}{2}(1-\gamma)} \quad (17)$$

Let us still consider the example when $d = 2n = 3$ and $\alpha_2 = \alpha_3 = 1$ in eq.(1). Then, the propagator behaves as $|x_1 - y_1|^{-1}$ for $x_2 - y_2 = x_3 - y_3 = 0$ and if $x_1 = y_1$ then the short distance behaviour is $|\mathbf{x} - \mathbf{y}|^{\gamma-1}$. We conclude that (if $d \leq 4$) increasing singularity of G leads to increasing regularity of the Feynman's causal propagator. However, if $\gamma < \frac{1}{2}$ then except of the case $d = 2n = 2$, the propagators remain singular (contrary to the suggestion of refs. [4]-[5]). By a formal argument, when $d = 2n = 4$, we can set $\gamma = 1$ in eqs.(7) and (17) with a conclusion that the singularity disappears. However, the case $\gamma = 1$ requires a careful examination because the integral (8) becomes divergent. For the general metric $g_{\mu\nu}$ in eq.(12) we can perform only perturbative calculations. We put $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We apply the stochastic representation (3) solving the corresponding stochastic equations for q_τ (see [6][8]) till the first order in h . We assume that h is Gaussian with the singularity $(x^2)^{-\gamma}$ on the light cone. These assumptions give the short distance behaviour $((x - y)^2)^{-n+1+\frac{1}{2}(2-n)\gamma/(1-\gamma)}$. It agrees with eq.(16), i.e., when all the coordinates relevant for the asymptotics are disturbed by a fluctuating metric.

In quantum field theory we obtain an average of many Green's functions. This average value has a functional representation similar to (7) (there will be a larger number of proper times τ). We can obtain the same conclusion that the average is less singular than in quantum field theory with a fixed metric.

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